



TITLE:

Commutators of Singular Integral Operators on Morrey Spaces with General Growth Functions (Harmonic Analysis and Nonlinear Partial Differential Equations)

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CITATION:

Mizuhara, Takahiro. Commutators of Singular Integral Operators on Morrey Spaces with General Growth Functions (Harmonic Analysis and Nonlinear Partial Differential Equations). 数理解析研究所講究録 1999, 1102: 49-63

ISSUE DATE:

1999-06

URL:

<http://hdl.handle.net/2433/63185>

RIGHT:

Commutators of Singular Integral Operators on Morrey Spaces with General Growth Functions

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Abstract of the Talk

The talk will be concerned with the boundedness of the commutators of Calderon-Zygmund singular integral operators on Morrey spaces $L^{p,\Phi}(R^n)$ with growth functions $\Phi(x, r)$ satisfying the condition ; there exists a constant C , independent of $(x, r) \in R_+^{n+1}$, such that for any $(x, r) \in R_+^{n+1}$

$$(1) \quad \int_r^\infty [\Phi(x, t)/t^{a+1}]dt \leq C\Phi(x, r)/r^a, \quad \text{for some } a > 0.$$

In this case, we write $\Phi \in G_a$ simply. We denote by $L^{p,\Phi}(R^n)$, $0 < p < \infty$, the space of locally integrable functions f , defined on R^n , for which there exists a constant C , independent of balls $B = B(x, r)$, such that

$$(2) \quad \int_{B(x,r)} |f(y)|^p dy \leq C^p \Phi(x, r)$$

for all balls $B = B(x, r)$. Let $\|f\|_{p,\Phi}$ be the smallest constant C satisfying (2). Then the space $L^{p,\Phi}(R^n)$ becomes a quasi-Banach space with quasi-norm $\|\cdot\|_{p,\Phi}$. In particular, if $1 \leq p < \infty$, then the space $L^{p,\Phi}(R^n)$ becomes a Banach space with norm $\|\cdot\|_{p,\Phi}$.

Let $BMO(R^n)$ be the space of all functions of bounded mean oscillation and let $\Lambda_\alpha(R^n)$, $0 < \alpha < n$, be the space of all Lipschitz continuous functions of order α . Let M be the Hardy-Littlewood maximal operator. We need two variants of M . For $0 < q < \infty$ let $M_q f(x) = \{(M|f|^q)(x)\}^{1/q}$. The sharp maximal function $f^\sharp(x)$ is defined by

$$f^\sharp(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y) - f_B| dy, \quad \text{where } f_B = |B|^{-1} \int_B f(y) dy.$$

Let T be a Calderon-Zygmund singular integral operator defined by $Tf = k * f$ with the kernel k satisfying the conditions :

$$\|\hat{k}\|_\infty \leq C, \quad |k(x)| \leq C|x|^{-n} \quad \text{for } 0 \neq x \in R^n,$$

$$|k(x) - k(x-y)| \leq C|y|/|x|^{n+1} \quad \text{for } |y| \leq |x|/2.$$

Let I_α , $0 < \alpha < n$, be the Riesz potential of order α defined by

$$(I_\alpha f)(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Related to $I_\alpha f$, the fractional maximal function $M_{\alpha/n}^* f(x)$ is defined by

$$M_{\alpha/n}^* f(x) = f_{\alpha,1}^*(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.$$

For a locally integrable function b and an operator S , we define the commutator $[b, S]$, between the operator S and the multiplication operator by b , by $[b, S] = bS - Sb$.

We have proved the following ([Miz₂]);

Theorem 1(Theorem 2.1.) *Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C = C(p, \Phi) > 0$, independent of f , such that for all $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(3) \quad \|Mf\|_{p,\Phi} \leq C \|f^\sharp\|_{p,\Phi}$$

where $L_c^\infty(R^n)$ be the set of all essentially bounded functions on R^n with compact support.

We use the method due to Di Fazio and Ragusa. Our method is based on weighted maximal inequality due to Garcia-Cuerva and Rubio de Francia.

From this and the pointwise estimate due to Strömberg ;

$$\{[b, T](f)\}^\sharp(x) \leq C \|b\|_* \{M_q(Tf)(x) + (M_s f)(x)\}, \quad 1 < q, s < \infty,$$

for almost all $x \in R^n$, we obtain the boundedness of the commutators $[b, T]$ on Morrey spaces ([Miz₂]) ;

Theorem 2(Theorem 2.2.) *Let $1 < p < \infty$ and $b \in BMO(R^n)$. We assume that $\Phi \in G_n$. Then the commutator $[b, T]$ is bounded in $L^{p,\Phi}$. More precisely, there exists a constant $C = C(p, \Phi) > 0$, independent of b and f , such that for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(4) \quad \|[b, T](f)\|_{p,\Phi} \leq C \|b\|_* \|f\|_{p,\Phi}.$$

Also we can observe the following ([Miz₂]) ;

Theorem 3(Theorem 2.3.) *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, T]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of b and f , such that for all $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(5) \quad \|[b, T]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha(R^n)} \|f\|_{p,\Phi}.$$

This follows from the result (due to Naki [N]) of the boundedness of Riesz potential on Morrey spaces and the pointwise estimate ;

$$|([b, T]f)(x)| \leq \int_{R^n} |b(x) - b(y)| |k(x-y)| |f(y)| dy \leq C \|b\|_{\Lambda_\alpha(R^n)} I_\alpha(|f|)(x).$$

Further we obtain the following result ([Miz₂]) from the boundedness of the fractional maximal operator $M_{\alpha/n}^*$ on Morrey spaces and the pointwise estimate due to Strömberg ;

$$\{[b, I_\alpha](f)\}^\sharp(x) \leq C \|b\|_* \{M_u(I_\alpha f)(x) + (M_{\alpha t/n}^* |f|^t)^{1/t}(x)\}$$

for almost all $x \in R^n$, where $1 < u, t < p < n/\alpha$.

Theorem 4(Theorem 3.1.) *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$ and $\Phi^{q/p} \in G_n$. If $b \in BMO(R^n)$, then the commutator $[b, I_\alpha]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of b and f , such that for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(6) \quad \|[b, I_\alpha]f\|_{q,\Phi^{q/p}} \leq C \|b\|_* \|f\|_{p,\Phi}.$$

Similarly we can show the following ([Miz₂]) ;

Theorem 5(Theorem 3.2.) *Let $1 < p < q < \infty$, $0 < \alpha, \beta$, $0 < \alpha + \beta = n(1/p - 1/q) < n$, $1 < p < n/(\alpha + \beta)$. We assume that $\Phi \in G_{n-p(\alpha+\beta)}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$. More precisely, there exists a constant $C = C(p, q, \Phi) > 0$, independent of b and f , such that for all $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi}(R^n) \cap L_c^\infty(R^n)$*

$$(7) \quad \|[b, I_\beta]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha(R^n)} \|f\|_{p,\Phi}.$$

Our results (Theorems 1, 2 and 4) generalize partly the classical results due to Di Fazio and Ragusa [DiFRag]. Also we obtain the new results (Theorems 3 and 5).

Acknowledgements. We wish to thank Professor Hideo Kozono for his kindness to invite me as a speaker of this Congress.

1. Introduction.

Let $\Phi = \Phi(x, r)$, be a growth function on $R_+^{n+1} = R^n \times R_+$, that is, a positive and non-decreasing function with respect to $r > 0$. We say that the growth function $\Phi(x, r)$ satisfies the Δ_2 -condition (or doubling condition) for $r > 0$ if there exists constant $D = D(\Phi) \geq 1$, independent of (x, r) , such that

$$(1.1) \quad \Phi(x, 2r) \leq D\Phi(x, r), \quad (x, r) \in R_+^{n+1},$$

or equivalently,

$$\Phi(x, 2r)/D \leq \Phi(x, r) \leq \Phi(x, 2r), \quad (x, r) \in R_+^{n+1}.$$

In this case, we write $\Phi \in \Delta_2$ simply. We consider the following functions in Δ_2 ;

$$\Phi(x, r) = \Psi(x)r^\lambda \{\log(1+r)\}^\mu, \quad \Psi(x) \in L^\infty(R^n), \quad 0 \leq \lambda < \infty, \quad -\infty < \mu < \infty.$$

Remark. Nakai [Nak] assumed a slightly weak condition on $\Phi(x, r)$ replacing (1.1) ; there exists a constant $C > 0$ such that, for all $(x, r) \in R_+^{n+1}$,

$$(1.2) \quad r \leq t \leq 2r \implies C^{-1} \leq \Phi(x, t)/\Phi(x, r) \leq C.$$

However, for simplicity, we describe the results on the assumption of (1.1). Of course our results are also valid under the condition (1.2).

Function Spaces. Let R^n be the n -dimensional Euclidean space and let $B = B(x, r)$ be the ball centered at $x \in R^n$ and with radius $r > 0$. Let $Q = Q(x, r)$ be the cube centered at $x \in R^n$ and with sides of length $r > 0$, where the cube will always mean a compact cube with sides parallel to the axes and nonempty interior. $|B|$ and $|Q|$ stand for the Lebesgue measures of ball B and cube Q , respectively. Let $0 < p < \infty$.

Definition 1.1 (Morrey spaces). (Confer Mizuhara [Miz₁]). We denote by $L^{p,\Phi} = L^{p,\Phi}(R^n)$ the space of locally integrable functions f , defined on R^n , for which there exists a constant C , independent of balls $B = B(x, r)$, such that

$$(1.3) \quad \int_{B(x,r)} |f(y)|^p dy \leq C^p \Phi(x, r)$$

for all balls $B = B(x, r)$. Let $\|f\|_{p,\Phi}$ be the smallest constant C satisfying (1.3). Then the space $L^{p,\Phi}$ becomes a quasi-Banach space with quasi-norm $\|\cdot\|_{p,\Phi}$ in the sense of Triebel [Tri]. In particular, if $1 \leq p < \infty$, then the space $L^{p,\Phi}$ becomes a Banach space with norm $\|\cdot\|_{p,\Phi}$. The balls $B = B(x, r)$ in (1.3) can be replaced by cubes $Q = Q(x, r)$.

When $\Phi(x, r) = r^\lambda$, $\lambda \geq 0$, then $L^{p, \lambda}$ is the classical Morrey space denoted by $L^{p, \lambda}$ simply. The classical Morrey spaces $L^{p, \lambda}$, $0 < \lambda < n$, were originally introduced by Morrey [Mor] in 1938 and used by himself and the others in the problems related to the calculus of variations and the theory of elliptic PDE's. We refer to Campanato [Cam], Giaginta [Gia], Kufner-John-Fučík [KufJohFuc] and Peetre [P₂].

The some properties of $L^{p, \lambda}$ are known ; If $1 \leq p < \infty$, then $L^{p, 0} = L^p(R^n)$ and $L^{p, n} = L^\infty(R^n)$ isometrically. If $n < \lambda$, then $L^{p, \lambda} = \{0\}$. If $1 \leq p < \infty$ and $0 < \lambda < n$, then $L^{p, \lambda}$ does not include nonzero constants. Hence, in the classical Morrey spaces, $L^{p, \lambda}$ for $0 < \lambda < n$ is interesting. Also Hölder's inequality implies the imbedding theorem ; if $(n - \lambda)/q = (n - \mu)/p$, $p \leq q$, then $L^{q, \lambda} \subset L^{p, \mu}$.

Let $BMO(R^n)$ be the John-Nirenberg space of all functions of bounded mean oscillation (see John-Nirenberg [JoN]), that is, $BMO(R^n)$ is a Banach space, modulo constants, with norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_B |B|^{-1} \int_B |b(y) - b_B| dy, \quad \text{where } b_B = |B|^{-1} \int_B b(y) dy.$$

The space $BMO(R^n)$ is identified with the dual space of the Hardy space $H^1(R^n)$ in the sense of Fefferman-Stein ([FeS₂]).

Let $\Lambda_\alpha(R^n)$, $0 < \alpha < n$, be the space of all Lipschitz continuous functions of order α on R^n . The space $\Lambda_\alpha(R^n)$ is homogeneous in the sense of dilations. The dual space of $H^p(R^n)$ can be identified with the Lipschitz space $\Lambda_\alpha(R^n)$, $\alpha = n(1/p - 1)$.

Classical operators. Let f be a locally integrable function on R^n . The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy$$

where the supremum is taken over all balls B containing x and $|B|$ is the volume of the ball B . We introduce two variants of M . Let $0 < q < \infty$ and

$$M_q f(x) = \{(M|f|^q)(x)\}^{1/q}.$$

Then Hölder's inequality shows that $Mf = M_1 f \leq M_q f$ if $1 \leq q < \infty$ and $M_q f \leq M_1 f = Mf$ if $0 < q \leq 1$. The sharp maximal function $f^\sharp(x)$ is defined by

$$f^\sharp(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y) - f_B| dy, \quad \text{where } f_B = |B|^{-1} \int_B f(y) dy.$$

Let T be a Calderon-Zygmund singular integral operator $Tf = k * f$ defined by the kernel k satisfying the conditions;

$$\|\hat{k}\|_{\infty} \leq C, \quad |k(x)| \leq C|x|^{-n} \quad \text{for } 0 \neq x \in R^n,$$

$$|k(x) - k(x - y)| \leq C|y|/|x|^{n+1} \quad \text{for } |y| \leq |x|/2.$$

For $\epsilon > 0$, put

$$T_{\epsilon}f(x) = \int_{|y|>\epsilon} k(y)f(x-y)dy \quad \text{and} \quad T^*f(x) = \sup_{\epsilon>0} |T_{\epsilon}f(x)|.$$

Let I_{α} , $0 < \alpha < n$, be the fractional integral operator (or Riesz potential operator) of order α defined by

$$(I_{\alpha}f)(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for a suitable function f . Related to $I_{\alpha}f$, the fractional maximal function $M_{\alpha/n}^*f(x)$, which appeared in [MucWhe] as $f_{\alpha,1}^*(x)$, is defined by

$$M_{\alpha/n}^*f(x) = f_{\alpha,1}^*(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.$$

We define the commutator $[b, S]$ between an operator S and the multiplication operator by a locally integrable function b , by $[b, S] = bS - Sb$.

In this note we show the boundedness of the commutator $[b, T]$, for $b \in BMO(R^n)$ or $b \in \Lambda_{\alpha}(R^n)$, on Morrey spaces $L^{p,\Phi}(R^n)$ with some growth function Φ . Our results (Theorems 2.1, 2.2, 3.1) generalize partly the recent results due to Di Fazio and Ragusa [DiFRag] on the classical Morrey spaces $L^{p,\lambda}(R^n)$, $0 < \lambda < n$, $1 < p < \infty$. Further we obtain the new results (Theorems 2.3 and 3.2). The letters C 's will denote positive constants, which may have different values in each line.

2. Commutators between Calderon-Zygmund singular integral operators and multiplication operator by a function $b \in BMO(R^n) \cup \Lambda_{\alpha}(R^n)$.

G_{α} -condition. We consider the following condition on growth function $\Phi(x, r)$;

$$\frac{\Phi(x, t)}{t^{\alpha}} \in L^1([r, \infty), dt/t)$$

for all $r > 0$ and any $x \in R^n$, and, in addition, there exists a constant C , independent of $(x, r) \in R_+^{n+1}$, such that

$$(2.1) \quad \int_r^{\infty} [\Phi(x, t)/t^{\alpha+1}] dt \leq C\Phi(x, r)/r^{\alpha}, \quad (x, r) \in R_+^{n+1},$$

for some $a > 0$. In this case, we write $\Phi \in G_a$ simply.

We can observe the following property of G_a , $a > 0$;

Lemma 2.1. (i) If $0 < a < a' < n$, then $G_a \subset G_{a'} \subset G_n \subset \Delta_2$.

(ii) If $\Phi \in \Delta_2$ with doubling constant D , $1 \leq D < 2^n$, then $\Phi \in G_n$.

(iii) If $a > 0$, then $G_a \subset G_{a\gamma}$ for some γ , $0 < \gamma < 1$. More precisely, if $\Phi \in G_a$, $a > 0$, there exist constants $\gamma = \gamma(C, a)$, $0 < \gamma < 1$, and $C' = C'(C, a, \gamma) > 0$ such that for any $(x, r) \in R_+^{n+1}$

$$(2.2) \quad \int_r^\infty [\Phi(x, t)/t^{a\gamma+1}] dt \leq C' \Phi(x, r)/r^{a\gamma}.$$

Proof. (i), (ii) These are easy to see.

(iii) Let

$$\Phi_a(x, r) = \int_r^\infty [\Phi(x, t)/t^{a+1}] dt.$$

Then (2.1) implies

$$\Phi_a(x, r) \leq C \Phi(x, r)/r^a.$$

For $0 < r < R$, we have, integrating by parts and using (2.1),

$$\begin{aligned} \int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt &= \int_r^R [\Phi(x, t)/t^{a+1} t^{a(\gamma-1)}] dt \\ &= [-\Phi_a(x, t) t^{a(1-\gamma)}]_r^R - \int_r^R [-\Phi_a(x, t) a(1-\gamma) t^{a(1-\gamma)-1}] dt \\ &= -\Phi_a(x, R) R^{a(1-\gamma)} + \Phi_a(x, r) r^{a(1-\gamma)} + a(1-\gamma) \int_r^R [\Phi_a(x, t) t^{a(1-\gamma)-1}] dt \\ &\leq C \Phi(x, r)/r^{a\gamma} + a(1-\gamma) C \int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt. \end{aligned}$$

Hence we obtain

$$\int_r^R [\Phi(x, t)/t^{a\gamma+1}] dt \leq \frac{C}{1 - a(1-\gamma)C} \Phi(x, r)/r^{a\gamma},$$

and we have (2.2) with

$$C' = \frac{C}{1 - a(1-\gamma)C} > 0.$$

Thus we have (2.2) for some γ such that $1 - (1/aC) < \gamma < 1$.

Q.E.D.

First using this Lemma, we show the following ;

Theorem 2.1. *Let $0 < p < \infty$. We assume that $\Phi \in G_n$. Then there exists a constant $C > 0$, independent of f , such that*

$$(2.3) \quad \|Mf\|_{p,\Phi} \leq C \|f^\sharp\|_{p,\Phi}$$

for all $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$, where $L_c^\infty(R^n)$ is the set of all essentially bounded functions on R^n with compact support.

Proof. We use the method due to Di Fazio-Ragusa [DifRag]. We recall the weighted version of the maximal inequality due to Fefferman-Stein [FS₂]; there exists a constant C such that

$$(2.4) \quad \int_{R^n} \{Mf(x)\}^p w(x) dx \leq C \int_{R^n} \{f^\sharp(x)\}^p w(x) dx$$

for all $w \in A_\infty$ and all $f \in L_w^p(R^n)$, for $0 < p < \infty$ (see Garcia-Cuerva-Rubio de Francia [GarRub; p.410]) where A_q , $1 \leq q \leq \infty$, is the Muckenhoupt class of weight functions.

Let $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$ and B a ball. We take $w(x)$ as $(M\chi)^\gamma \in A_1$, $0 < \gamma < 1$, where $\chi = \chi_B(x)$ is the characteristic function of the ball $B = B(x_0, r)$.

Then we get by (2.4),

$$\begin{aligned} \int_B \{Mf(x)\}^p dx &= \int_{R^n} \{Mf(x)\}^p \chi_B(x) dx \\ &\leq \int_{R^n} \{Mf(x)\}^p \{M\chi_B(x)\}^\gamma dx \leq C \int_{R^n} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx \\ &= C \int_{B(x_0,r)} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx \\ &\quad + C \sum_{k=1}^{\infty} \int_{B(x_0,2^k r) - B(x_0,2^{k-1}r)} \{f^\sharp(x)\}^p \{M\chi_{B(x_0,r)}(x)\}^\gamma dx \\ &\leq C \left\{ \int_{B(x_0,r)} \{f^\sharp(x)\}^p dx + \sum_{k=1}^{\infty} (2^{-k n \gamma}) \int_{B(x_0,2^k r)} \{f^\sharp(x)\}^p dx \right\} \\ &\leq C \|f^\sharp\|_{p,\Phi}^p \left\{ \Phi(x_0, r) + \sum_{k=1}^{\infty} (2^{-k})^{n\gamma} \Phi(x_0, 2^k r) \right\} \\ &\leq C \|f^\sharp\|_{p,\Phi}^p \sum_{k=0}^{\infty} \frac{\Phi(x_0, 2^k r)}{2^{kn\gamma}} \sim C \|f^\sharp\|_{p,\Phi}^p r^{n\gamma} \int_r^\infty \frac{\Phi(x_0, t)}{t^{n\gamma+1}} dt. \end{aligned}$$

Since, by Lemma 2.1, the last term is bounded by

$$C \|f^\sharp\|_{p,\Phi}^p \Phi(x_0, r),$$

we have

$$\|Mf\|_{p,\Phi} \leq C\|f^\sharp\|_{p,\Phi}.$$

Thus we have (2.3) for some $C > 0$, independent of $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

Our second aim is to show the following ;

Theorem 2.2. *Let $1 < p < \infty$, $b \in BMO(R^n)$ and T be a Calderon-Zygmund singular integral operator. We assume that $\Phi \in G_n$. Then the commutator $[b, T]$ is bounded in $L^{p,\Phi}$. More precisely, there exists constant C , independent of b and f , such that*

$$(2.5) \quad \|[b, T](f)\|_{p,\Phi} \leq C\|b\|_*\|f\|_{p,\Phi}$$

for all $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem we need Theorem 2.1 and the following three lemmas ;

Lemma 2.2. *Let $1 < q, s < \infty$, $b \in BMO(R^n)$ and T be a Calderon-Zygmund singular integral operator. Then there exists constant C independent of b and f such that*

$$\{[b, T](f)\}^\sharp(x) \leq C\|b\|_*\{M_q(Tf)(x) + (M_sf)(x)\}$$

for almost all $x \in R^n$ and all $f \in L_c^\infty(R^n)$

Proof. This is the pointwise estimate due to Strömberg (see [Tor, p.418.] and Janson [Jan; pp.268-269.]).

Q.E.D.

Lemma 2.3. *Let $0 < q < p < \infty$. We assume that $\Phi \in G_n$. Then the maximal operator M_q is a bounded operator in $L^{p,\Phi}(R^n)$ and*

$$\|M_q f\|_{p,\Phi} \leq C\|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}(R^n)$.

Proof. The proof depends on the weighted maximal inequality due to Fefferman-Stein [FefSte₁]. In the restricted case $1 \leq q < p < \infty$, the corresponding result is proved by Nakai [Nak; Theorem 1]. It is not difficult to extend the result to the case $0 < q < p < \infty$. Confer also Chiarenza-Frasca [ChiFra] and Mizuhara [Miz₁].

Q.E.D.

Lemma 2.4. *Let $1 < p < \infty$. We assume that $\Phi \in G_n$. Then the Calderon-Zygmund singular integral operator T is a bounded operator in $L^{p,\Phi}(R^n)$ and*

$$(2.6) \quad \|Tf\|_{p,\Phi} \leq C\|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}(R^n)$.

Proof. This is the result due to Nakai [Nak; Theorem 2] in the setting of more general growth functions. Confer also Peetre [Pee₁], Chiarenza-Frasca [ChiFra] and Mizuhara [Miz₁].

We note that we can give a short proof following the method of the author [Miz₁] which depends on the weighted maximal inequality due to Cordoba-Fefferman [CorFef] (see also [GarRub]);

there exists constant C , depending only on T, p and $0 < \gamma < 1$, such that

$$(2.7) \quad \int_{R^n} |Tf(x)|^p \phi(x) dx \leq C \int_{R^n} |f(x)|^p (M\phi)^\gamma(x) dx$$

for all f and $\phi(x) \geq 0$. A standard proof using (2.7) implies (2.6).

Q.E.D.

Proof of Theorem 2.2. We apply the method of Di Fazio-Ragusa [DifRag] to our case. We suppose that $b \in BMO(R^n)$. Then Theorem 2.1 and Lemma 2.2 imply that, for $1 < q, s < p < \infty$,

$$\begin{aligned} \|[b, T](f)\|_{p,\Phi} &\leq \|M\{[b, T](f)\}\|_{p,\Phi} \\ &\leq C\|([b, T](f))^\sharp\|_{p,\Phi} \leq C\|b\|_* \{ \|M_q(Tf)\|_{p,\Phi} + \|M_s f\|_{p,\Phi} \}. \end{aligned}$$

Since, Lemma 2.3 and Lemma 2.4 imply

$$\|M_q(Tf)\|_{p,\Phi} \leq C\|Tf\|_{p,\Phi} \leq C\|f\|_{p,\Phi} \quad \text{and} \quad \|M_s f\|_{p,\Phi} \leq C\|f\|_{p,\Phi},$$

we obtain

$$\|[b, T](f)\|_{p,\Phi} \leq C\|b\|_* \|f\|_{p,\Phi}$$

for $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$. Thus we have (2.5).

Q.E.D.

When $b \in \Lambda_\alpha(R^n)$, $0 < \alpha < n$, we obtain the following :

Theorem 2.3. Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, T]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

$$(2.8) \quad \|[b, T]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha(R^n)} \|f\|_{p,\Phi}$$

for some constant C independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem we need the following lemma ;

Lemma 2.5. Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. Then the fractional integral operator I_α is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

$$(2.9) \quad \|I_\alpha f\|_{q,\Phi^{q/p}} \leq C \|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}$.

Proof. This is the result due to Nakai [Nak; Theorem 3].

Q.E.D.

Proof of Theorem 2.3. Let $b \in \Lambda_\alpha(R^n)$. Then

$$\begin{aligned} |([b, T]f)(x)| &\leq \int_{R^n} |b(x) - b(y)| |k(x-y)| |f(y)| dy \\ &\leq C \|b\|_{\Lambda_\alpha} \int_{R^n} |x-y|^\alpha |x-y|^{-n} |f(y)| dy = C \|b\|_{\Lambda_\alpha} I_\alpha(|f|)(x). \end{aligned}$$

Hence we have, by Lemma 2.5,

$$\|[b, T]f\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha} \|I_\alpha(|f|)\|_{q,\Phi^{q/p}} \leq C \|b\|_{\Lambda_\alpha} \|f\|_{p,\Phi}.$$

Thus we have (2.8) for some $C > 0$, independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

3. Commutators between the fractional integral operator and multiplication operator by a function $b \in BMO(R^n) \cup \Lambda_\alpha(R^n)$.

In this section first we show the following when $b \in BMO(R^n)$;

Theorem 3.1. *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$ and $\Phi^{q/p} \in G_n$. If $b \in BMO(R^n)$, then the commutator $[b, I_\alpha]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and*

$$(3.1) \quad \|[b, I_\alpha]f\|_{q,\Phi^{q/p}} \leq C \|b\|_* \|f\|_{p,\Phi}$$

for some constant C independent of $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

To prove the theorem, we need the following pointwise estimate and the bounbedness of the fractional maximal operator ;

Lemma 3.1. *Let $0 < \alpha < n$, $1 < u$, $t < n/\alpha$ and $b \in BMO(R^n)$. Then there exists constant C independent of b and f such that*

$$\{[b, I_\alpha](f)\}^\sharp(x) \leq C \|b\|_* \{M_u(I_\alpha f)(x) + (M_{\alpha/n}^* |f|^t)^{1/t}(x)\}$$

for almost all $x \in R^n$ and all $f \in L_c^\infty(R^n)$

Proof. This is the pointwise estimate due to Strömberg (see [Tor, p.419.] and Di Fazio-Ragusa [DifRag; p.326, Lemma 2.]).

Q.E.D.

Lemma 3.2. *Let $1 < p < q < \infty$, $0 < \alpha = n(1/p - 1/q) < n$. We assume that $\Phi \in G_{n-p\alpha}$. Then the fractional maximal operator $M_{\alpha/n}^*$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and*

$$(3.2) \quad \|M_{\alpha/n}^* f\|_{q,\Phi^{q/p}} \leq C \|f\|_{p,\Phi}$$

for some constant C independent of $f \in L^{p,\Phi}$.

Proof. Let $B(z, r)$ be any ball centered at z and with radius $r > 0$ such that $x \in B(z, r)$. Since

$$\begin{aligned} I_\alpha(|f|)(x) &:= \int_{R^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \geq \int_{B(z,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \\ &\geq \frac{C'}{r^{n-\alpha}} \int_{B(z,r)} |f(y)| dy \simeq \frac{C''}{|B|^{1-\alpha/n}} \int_{B(z,r)} |f(y)| dy, \end{aligned}$$

we have the pointwise estimate ;

$$(M_{\alpha/n}^* f)(x) \leq I_\alpha(|f|)(x)$$

for almost all $x \in R^n$ and all $f \in L^{p,\Phi}$. Hence Lemma 2.5 implies the result.

Q.E.D.

Proof of Theorem 3.1. Let $b \in BMO(R^n)$. Then Theorem 2.1 and Lemma 3.1 imply that, for $1 < u, t < q < p < \infty$,

$$\begin{aligned} \| [b, I_\alpha](f) \|_{q,\Phi^{q/p}} &\leq \| M\{[b, I_\alpha](f)\} \|_{q,\Phi^{q/p}} \leq \| \{([b, I_\alpha](f))^\# \} \|_{q,\Phi^{q/p}} \\ &\leq C \| b \|_* \{ \| M_u(I_\alpha f) \|_{q,\Phi^{q/p}} + \| (M_{\alpha t/n}^* |f|^t)^{1/t} \|_{q,\Phi^{q/p}} \}. \end{aligned}$$

Also, under the assumption on Φ , Lemma 2.3 and Lemma 2.5 imply

$$\| M_u(I_\alpha f) \|_{q,\Phi^{q/p}} \leq C \| I_\alpha f \|_{q,\Phi^{q/p}} \leq C \| f \|_{p,\Phi}$$

and Lemma 3.2 imply

$$\begin{aligned} \| (M_{\alpha t/n}^* |f|^t)^{1/t} \|_{q,\Phi^{q/p}} &= \| M_{\alpha t/n}^* |f|^t \|_{q/t,\Phi^{q/p}}^{1/t} \\ &\leq C \| (|f|^t) \|_{p/t,\Phi}^{1/t} = C \| f \|_{p,\Phi}. \end{aligned}$$

Hence we obtain

$$\| [b, I_\alpha](f) \|_{q,\Phi^{q/p}} \leq C \| b \|_* \| f \|_{p,\Phi}$$

for $b \in BMO(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$. Thus we have (3.1).

Q.E.D.

We close this section showing the following ;

Theorem 3.2. Let $1 < p < q < \infty$, $0 < \alpha, \beta$, $0 < \alpha + \beta = n(1/p - 1/q) < n$, $1 < p < n/(\alpha + \beta)$. We assume that $\Phi \in G_{n-p(\alpha+\beta)}$. If $b \in \Lambda_\alpha(R^n)$, then the commutator $[b, I_\beta]$ is a bounded operator from $L^{p,\Phi}(R^n)$ into $L^{q,\Phi^{q/p}}(R^n)$ and

$$(3.3) \quad \| [b, I_\beta]f \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| f \|_{p,\Phi}$$

for some constant C independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Proof of Theorem 3.2. Let $b \in \Lambda_\alpha(R^n)$. Then

$$|([b, I_\beta]f)(x)| \leq C \| b \|_{\Lambda_\alpha} I_{\alpha+\beta}(|f|)(x).$$

for almost all $x \in R^n$. Hence we have, by Lemma 2.5,

$$\| [b, I_\beta]f \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| I_{\alpha+\beta}(|f|) \|_{q,\Phi^{q/p}} \leq C \| b \|_{\Lambda_\alpha} \| f \|_{p,\Phi}.$$

Thus we have (3.3) for some $C > 0$, independent of $b \in \Lambda_\alpha(R^n)$ and $f \in L^{p,\Phi} \cap L_c^\infty(R^n)$.

Q.E.D.

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